

# Critical Properties of an Ising Model with Dilute Long-range Interactions

R.T. Scalettar

*Physics Department, University of California, Davis, CA 95616*

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Statistical mechanical models with local interactions in  $d > 1$  dimension can be regarded as  $d = 1$  dimensional models with regular long range interactions. In this paper we study the critical properties of Ising models having  $V$  sites, each having  $z$  randomly chosen neighbors. For  $z = 2$  the model reduces to the  $d = 1$  Ising model. For  $z = \infty$  we get a mean field model. We find that for finite  $z > 2$  the system has a second order phase transition characterized by a length scale  $L = \ln V$  and mean field critical exponents that are independent of  $z$ .

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The dimensionality of a system and the symmetries of its Hamiltonian determine its critical behavior, which is characterized by the exponents that describe the singularities in magnetization, susceptibility, and specific heat at the critical temperature  $T_c$ . Meanwhile, features of a model such as additional terms in the Hamiltonian of the same dimensionality and symmetry are irrelevant to these properties. Such “universality” applies, however, only to systems with the same range of interactions. Exponents are affected by long range forces which can alter the way correlations diverge at the critical temperature.

As a particular instance of this effect, increasing the dimensionality can be regarded as the addition of peculiar long range interactions: The  $d = 2$  Ising model on a  $N \times N$  lattice is trivially equivalent to a “ $d = 1$ ” model where site  $l$  connects to sites  $l \pm \hat{x}$  and  $l \pm N\hat{x}$ . This new model is characterized by a coordination number  $z = 4$  and a regular set of long range bonds. The critical exponents were changed by the addition of the  $l \pm N\hat{x}$  bonds. In order to study this effect, and in particular to understand the importance of the regularity of the additional connections, we consider in this paper a “ $d = 1$ ” ferromagnetic Ising model

$$H = -J \sum_{\langle ij \rangle} S_i S_j \quad (1)$$

where  $S_i$  is a classical variable which can take the values  $\pm 1$  and the sum is over “random” neighbors  $\langle ij \rangle$  defined more precisely as follows. Suppose we have a total of  $V$  sites each of which is to have  $z$  randomly chosen neighbors. Our operational prescription for the construction of random neighbors is to begin with a list of integers of length  $zV$  in which the first  $V$  positive integers are repeated  $z$  times. This list is randomized by a large number of pair interchanges. Sites are defined to be neighbors if they appear as the  $2n - 1$  and  $2n$  entries in the list,  $n = 1, 2, \dots, \frac{zV}{2}$ . We eliminate by further randomization any pairs that appear twice or sites that are self-connected. The end result of this procedure is a lattice in which each spin has  $z$  neighbors randomly chosen from the  $V - 1$  remaining spins[1].

we will construct the  $d = 1$  dimensional Ising model, if we insist that the lattice have all sites in the same cluster. For  $z = 4$  we would have a model similar to the  $d = 2$  Ising Hamiltonian in the sense that each site would be connected to four neighbors, two of which could be arbitrarily considered as geometrically close, the other two of which are not. However, the disordered nature of the bonds has an important qualitative effect. The bonds are “more long ranged” than the regular  $d$ -dimensional case. Consider the average separation,  $\langle \sigma_{ij} \rangle_c$ , between two spins, defined as the average over all pairs of spins of the minimum number of bonds traversed in moving from one to another. (The subscript “c” is included to emphasize that this is an average over the different pairs of sites in the lattice rather than the usual thermodynamic average. We could also ask about an average over different possible realizations of the random bonds. However, we shall assume that in the thermodynamic limit of large  $V$  the lattice is self-averaging.) For short-ranged models in any dimension  $d$ , the average separation grows with the linear extent  $L$ . That is,  $\langle \sigma_{ij} \rangle \approx L = V^{\frac{1}{d}}$ . For the random bond case,  $\langle \sigma_{ij} \rangle = a(z) + b(z)\ln V$ . The average separation of spins is much smaller. The role of  $z$  is not fundamental in the sense that it controls only the values of  $a$  and  $b$ , not the logarithmic dependence on  $V$ . This result can be obtained numerically by explicit construction of lattices and computing  $\sigma_{ij}$ . In Fig. 1 we show such numerically obtained values for our random long range model.

The behavior of the average separation with volume can also be motivated analytically by considering a Bethe lattice of coordination number  $z$ . Such a lattice differs from that described above in having no closed loops, but nevertheless, as we will see in greater detail below, is relevant to the random bond model we are studying. The central site from which the lattice is grown has  $z$  first neighbors,  $z(z - 1)$  second neighbors, and  $z(z - 1)^{n-1}$  neighbors of distance  $n$ . If the lattice is terminated after level  $N$ , then the total number of sites is  $V = 1 + z \sum_{n=1}^N (z - 1)^{n-1}$  and the distance to the central site in the thermodynamic limit is  $\frac{1}{V} \sum_{n=1}^N (z - 1)^{n-1} = \frac{1}{V} \ln V$ . Thus, the Poreldis-

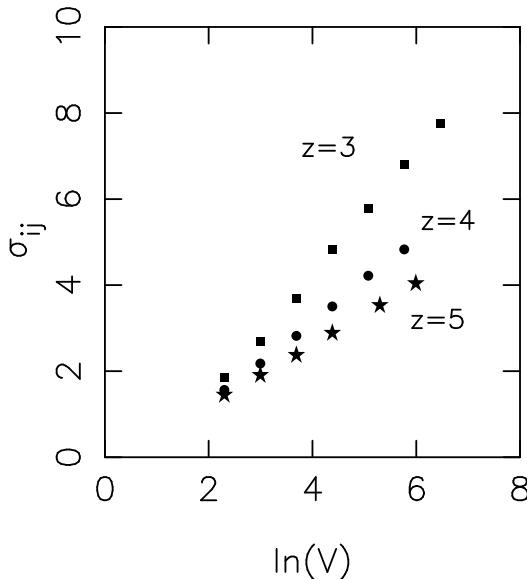


FIG. 1: The average separation between spins  $\langle \sigma_{ij} \rangle_c$  is shown as a function of  $\ln V$  for different choices of the coordination number  $z$ . The slope of the curve is  $b(z)$ .

tice of coordination number  $z$  exhibits a similar logarithmic growth of  $\langle \sigma_{ij} \rangle$  with  $V$  and has the particular value  $b(z) = \frac{1}{\ln(z-1)}$ .

This very slow growth, in a fashion qualitatively independent of  $z$  for  $z \geq 3$ , of the “distance” between bonds with the system size might suggest that the critical properties will be more similar to that of the mean field case, where the distance between all bonds is unity, than the case of finite dimension where  $\langle \sigma_{ij} \rangle \approx V^{\frac{1}{d}}$ . Indeed, the apparent absence of a length scale would also suggest mean field like behavior. This is in accordance with the fact that if one defines the dimensionality of a system[2] as  $d = \lim_{n \rightarrow \infty} [\ln c_n / \ln n]$ , where  $c_n$  is the number of sites within  $n$  steps of a given site, then one finds  $d = \infty$ . However, the number of paths between spins is also relevant in determining the critical behavior[?], and this number of paths does depend on  $z$ , so these arguments suggestive of mean field behavior are not conclusive.

To make this point more precise, we note that regular  $d$ -dimensional lattice Hamiltonians are characterized by the existence of short loops of bonds which connect sites back to themselves. In contrast, if the underlying lattice is a Cayley tree, sites are connected to neighbors without any such closed loops. The model discussed here can be viewed as an alternate way of tying off the ends of a Cayley tree lattice in which closed loops of size  $\ln V$  are created. For such structures it is known that the manner in which the boundary, which is a nonvanishing fraction of the total number of sites in the thermodynamic limit, is treated is important to the critical behavior[4].

Calculation of the energy cost for introducing a domain wall which divides the lattice in two is also instructive. Consider first the usual Ising model on a hypercubic lattice. For such a model, the critical temperature is given by

a single bond need be broken. In two dimensions  $V^{\frac{1}{2}}$  bonds must be broken, and in  $d$  dimensions we need to break  $V^{\frac{d-1}{d}}$  bonds. This increasing difficulty of forming a domain wall is, of course, why the tendency towards ordering is higher as the dimensionality increases. In the random bond model we have described, an attempt to divide the lattice in two equal pieces would on the average necessitate a breaking of  $\frac{V}{2}$  bonds, a scaling with the volume  $V$  which is the  $d \rightarrow \infty$  limit of the usual Ising case. This further points towards a mean field scenario for the critical properties.

We can analyze in a little more detail the effect of the behavior of  $\langle \sigma_{ij} \rangle$  by considering a high temperature expansion. To lowest order

$$\begin{aligned} \langle S_i S_j \rangle &\approx \tau^{\sigma_{ij}} \\ \tau &= \tanh(\beta J). \end{aligned} \quad (2)$$

(The Griffith’s inequality in fact guarantees that this is a lower bound on the correlation function[5].) In 1 dimension for free boundary conditions the shortest path is the *only* path and this expression is exact.  $\langle S_i S_j \rangle$  decays exponentially to zero with increasing separation except at  $T = T_c = 0$ .

The magnetic susceptibility is a sum over all such correlation functions. For temperature  $T > T_c$  where the magnetization is zero, the susceptibility per site is

$$\begin{aligned} \frac{1}{V} \chi &= \frac{1}{V} \sum_{ij} \langle S_i S_j \rangle \approx V \langle \tau^{\sigma_{ij}} \rangle_c \\ &\approx V \tau^{\langle \sigma_{ij} \rangle_c} = \tau^{a(z)} V^{1-b(z)|\ln(\tau)|}. \end{aligned} \quad (3)$$

Thus we estimate a critical temperature at which  $\frac{\chi}{V}$  diverges in the thermodynamic limit as

$$\tau_c = \tanh(\beta_c J) \approx e^{-\frac{1}{b(z)}}. \quad (4)$$

This argument, in fact, gives the exact result for the Bethe lattice, since if we substitute  $b(z) = \frac{1}{\ln(z-1)}$  into Eq. (4), we obtain  $\beta_c J = \frac{1}{2} \ln \frac{z}{z-1}$ , which is precisely the Bethe lattice critical temperature. On the other hand, within MFT,  $\frac{T_c}{z} = z$ . In Fig. 2 we show, as the full curve, a plot of  $\frac{T_c}{z}$  vs.  $z$  obtained from Eq. (4) using Bethe lattice values of  $b(z)$  which are close to those obtained numerically for the random bond prescription described above. The squares are obtained by scaling monte carlo results to obtain  $T_c$  as discussed shortly.

In order to see whether these arguments suggesting some sort of mean field theory really are valid, we have performed monte carlo simulations. In Fig. 3 we show results for the susceptibility per site,  $\frac{\chi}{V}$ , vs  $T$  for  $z = 3$  and 4. The different symbols represent lattices of different size, typically between  $V = 200$  and  $V = 6400$  sites. The increasing sharpness and height with volume indicate the presence of a finite  $T_c$  which increases with the coordination of the lattice. Fig. 4 shows similar results for the specific heat  $C$ . A comparison with

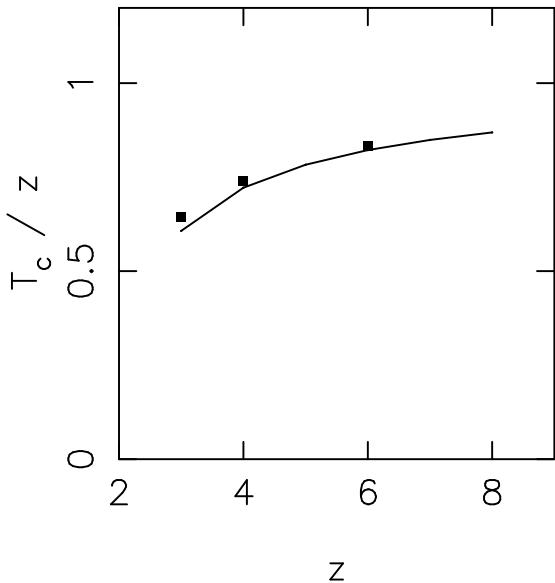


FIG. 2: The approximate analytic form for the critical temperature  $T_c$  obtained from Eq. (4) normalized by the coordination number  $z$ . The Bethe lattice form of  $b(z)$  has been used. The squares are values for the normalized critical temperature obtained from scaling finite size monte carlo results for the susceptibility and the specific heat.

nonanalytic behavior as  $V \rightarrow \infty$ . The values of  $T_c$  are consistent with those suggested by Fig. 3.

To explore the order of the phase transition, we can look at histograms of the magnetization distribution  $P(m)$  for different temperatures. At high temperatures  $P(m)$  exhibits a single peak at  $m = 0$ . As  $T$  is decreased this peak is broadened and eventually splits into two peaks symmetrically located about  $\pm m_0$ . At no temperature is evidence seen for the 3 peaked structure characteristic of a first order transition where  $m = 0$  and  $m \neq 0$  phases are in coexistence. This suggests the transition is second order for finite  $z$ , as it must be for  $z = \infty$ .

It is natural to attempt to perform finite size scaling on the data. We can first imagine the maxima in the plots of the susceptibility and specific heat represent values of “ $T_c$ ” on finite size lattices. Conventional scaling theory[6] suggests that

$$T_c(V) = T_c(\infty)[1 - L^{-\frac{1}{\nu}}]. \quad (5)$$

We are immediately confronted with the difficulty of relating the linear extent  $L$  to the number of sites  $V$ . Choosing  $L = V$  would in the usual d-dimensional models simply rescale the exponent  $\nu$  by the dimensionality. We find that to obtain a reasonable fit to this form we would need to use an anomalously large value of  $\nu$ . In fact, a better *ansatz* is obtained by replacing  $L$  by  $\ln V$ . We then find that the best fit for  $z = 3$  is obtained by  $T_c(\infty) = 1.91 \pm 0.05$  and  $\nu = 0.52 \pm 0.06$ . This is, of course, in reasonable agreement with the mean field value for the correlation length exponent  $\nu$ .

The situation for the more difficult  $z = 4$  is similar.

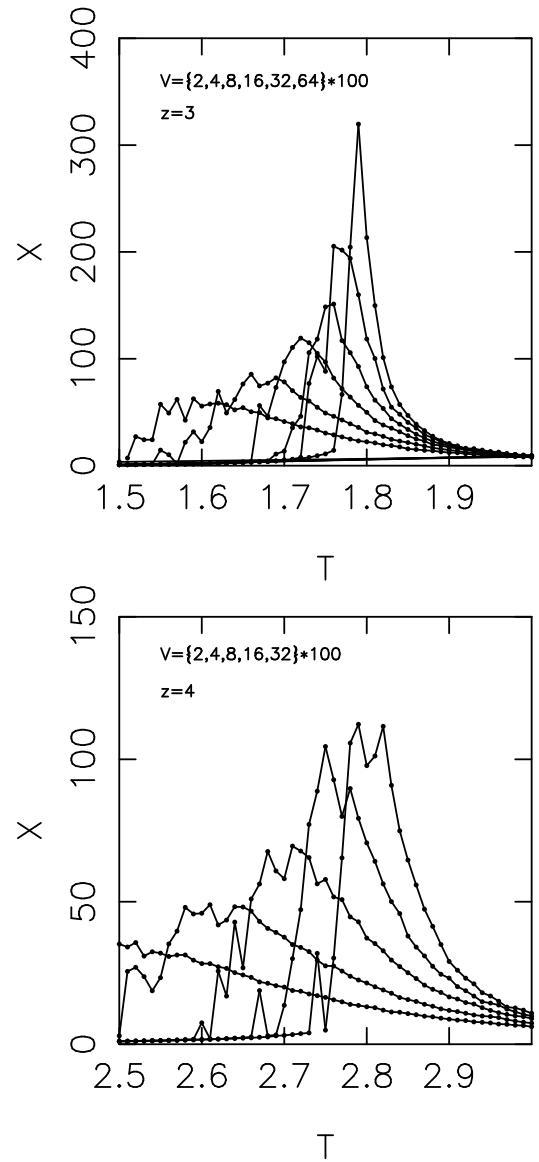


FIG. 3: Raw results for the susceptibility per site as a function of temperature for different sized lattices. Fig. 3a is for  $z = 3$  and Fig. 3b for  $z = 4$ .

$\chi$ , we expect that curves for different lattices, distinct in their unscaled forms of Figs. 3 and 4, will fall on universal curves when scaled with appropriate choices of the critical temperature and exponents[6]. That is, for  $\chi$

$$\frac{\chi}{L^{\frac{\gamma}{\nu}}} = f(L^{\frac{1}{\nu}}t), \quad (6)$$

while for  $C$

$$\frac{C}{L^{\frac{\alpha}{\nu}}} = g(L^{\frac{1}{\nu}}t). \quad (7)$$

Here  $f$  and  $g$  are some scaling functions. In Fig. 5 we show  $\chi$  scaled appropriately using  $L = \ln V$  and the critical exponents  $\nu = 0.53$  and  $\gamma = 1.06$ . We have chosen  $t = \ln V$  and  $t = 2^{-1/(1-\nu)}$  for  $z = 3$  and  $z = 4$  respectively.

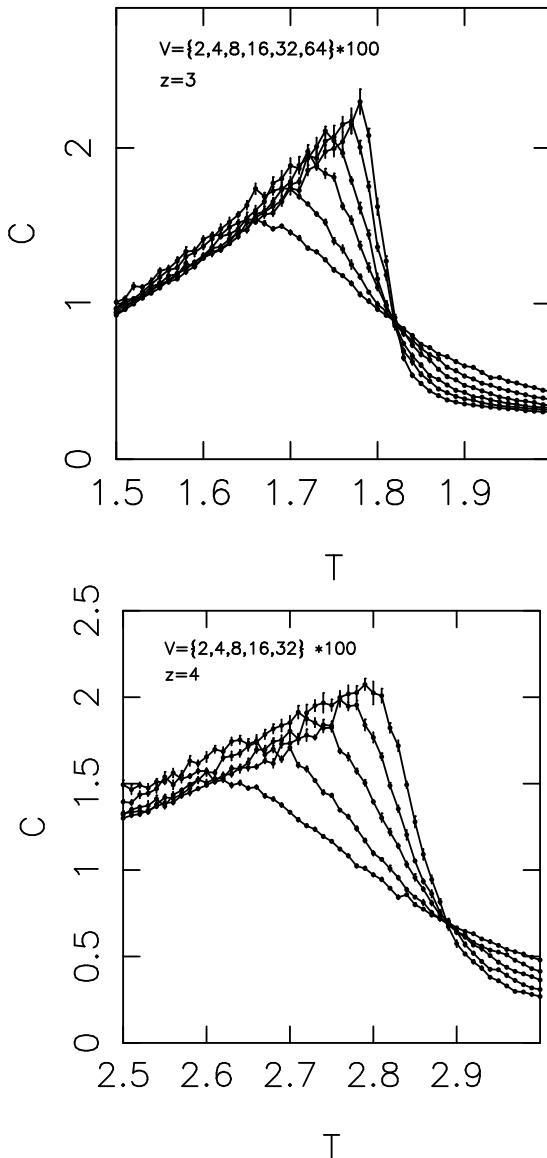


FIG. 4: Raw results for the specific heat per site as a function of temperature for different sized lattices. Fig. 4a is for  $z = 3$  and Fig. 4b for  $z = 4$ .

The critical temperature  $T_c = 1.87$ . This scaling plot has somewhat more scatter than the ones obtained for regular models[7]. However, this can be attributed to the need to average the results, especially for smaller lattices, over different configurations of random bonds. For the specific heat, we note that the maximum in  $C$  in Fig. 4 is insensitive to lattice size. This translates to a small value for the exponent  $\alpha$ , which is zero in the mean field limit. To estimate the accuracy of these values of the exponents, we have examined scaling plots in which the exponents have been shifted. We see a significant deterioration of the quality for changes of more than 10 percent.

We have seen that the Ising model defined by Eq. 1 and the resulting free field theory is a mean field theory.

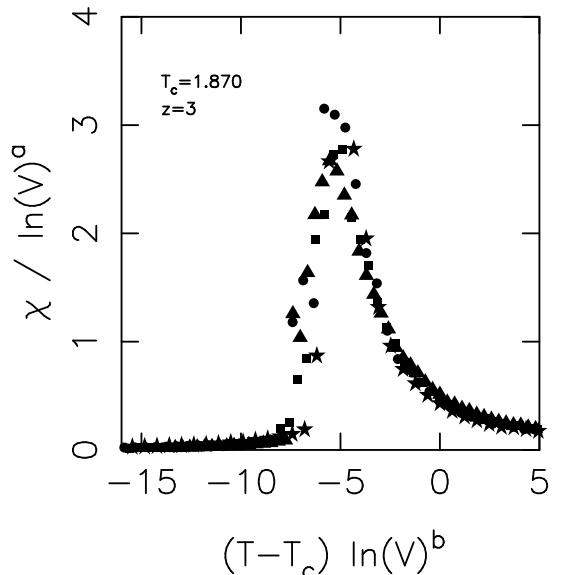


FIG. 5: The results of Fig. 3a are shown using the appropriate scaled forms as described in the text. The critical exponents used were  $\nu = 0.53$  and  $\gamma = 1.06$ . The critical temperature  $T_c = 1.87$ . Different symbols correspond to different lattice sizes,  $V = 800, 1600, 3200$ , and  $6400$ .

field like critical behavior with a length scale which depends logarithmically on the “volume” of the system. Clearly the regular  $d$ -dimensional models are realized in our model by certain (very) improbable choices of the random bonds. We can ask about the renormalization group flows in the space of possible choices of the bonds. If we construct some quantitative measure of the distance of a random assignment from the usual regular one, what will be the extent of the basin of attraction of the fixed points corresponding to the regular models and their exponents? Is it vanishingly small, so that in some sense *all* models of this sort flow towards mean field theory, or will there be some finite region characterized by the regular exponents?

Traditionally, one route to mean field theory is to increase the dimensionality of the system, which has the effect of enhancing the connectivity of the lattice while leaving the interactions short ranged. In a similar manner, mean field results are obtained for the critical properties of the Bethe lattice in the limit where the coordination  $z$  goes to infinity, but again the interaction is short ranged. Both these approaches, then, take a route to MFT via increased numbers of neighbors. However, we have seen here a novel realization which does not proceed via increased connectivity but rather works with fixed coordination and considers a model with extremely long ranged interactions. Kac et.al.[8] have shown in a one dimensional model with an interaction of length scale  $R$  that there is a nonzero  $T_c$  only in the  $R \rightarrow \infty$  limit. There, however,  $R$  and  $z$  are *simultaneously* increased. It is clear from our work that increasing  $R$  alone, at fixed  $z$ , leads to a mean field like transition. This is in contrast to the Bethe lattice case where the critical temperature is zero for any finite coordination  $z$ .

that there are two alternate routes to mean field critical behavior: either diverging  $z$  or diverging  $R$  alone suffice. As we have pointed out, this is perhaps not too surprising since neighbors in additional spatial dimensions can

equivalently be regarded as regular long range bonds.

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- [1] It is to be emphasized that the connections between sites  $i$  and  $j$  are symmetric ones, i.e. once it is determined that two sites are joined by a bond, then the Hamiltonian Eq. (1) describes the thermodynamics in the usual way. In contrast, within the context of studies of Kauffman genetic networks, Derrida et.al. have considered the dynamic evolution of systems of Ising spins with asymmetric connections in which spin  $i$  might influence the subsequent value of  $j$ , while  $j$  has no direct effect on  $i$ . They find that  $z = 2$  is a critical value of the connectivity in much the same way that in the present equilibrium statistical mechanical model there is no phase transition unless  $z > 2$ . B. Derrida and Y. Pomeau, *Europhys. Lett.* **1**, 45 (1986). Further discussion of networks with dilute, asymmetric bonds is contained in K.E. Kurten, *Phys. Lett.* **A129**, 157 (1989) and *J. Physique (France)* **51**, 1585 (1990).
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